Biorthogonality—Revisited
&
A Generalized Spectral Decomposition Theorem

Nicholas Wheeler
June 2016

Introduction. In the differential geometry of surfaces the 1st and 2nd fundamental forms give rise at each point on Σ to a pair of real symmetric 2×2 matrices, $\mathbb{G}$ and $\mathbb{H}$, of which the former—which provides a description of ds^2—is invariably positive-definite but the latter sometimes isn't. The local Gaussian curvature of Σ can be described

$$K = \frac{\det \mathbb{H}}{\det \mathbb{G}}$$

so at points of negative curvature $\mathbb{H}$ is in fact *negative*-definite. Familiarly, the eigenvalues of such matrices are necessarily real, and (in the absence of spectral degeneracy) the associated eigenvectors necessarily orthogonal. So naturally associated with every such matrix is an orthogonal "eigenbasis" in $\mathcal{V}_2$. But negative-definite matrices give rise also (as explained below) to an equally natural *non*-orthogonal basis, the elements of which in differential geometry indicate "self-conjugate" or "asymptotic" directions. It is thus from differential geometry that I have acquired an interest in the *representation of vectors and linear operators with respect to non-orthogonal bases*.

The discussion will bring into play the "dual" of any given non-orthogonal basis, and will establish the sense in which a basis and its dual are "biorthogonal." Accidental simplifications arise in the low-dimensional cases of highest practical interest; those are of interest in their own right, but tend to obscure the essentials of the general theory. I look initially therefore to the case $\mathcal{V}_n$, and only after the general principles are in place to the cases $\mathcal{V}_2$ and $\mathcal{V}_3$. I had occasion long ago to visit this general subject area. My motivation, emphasis and notation were on that occasion (where the arena was $\mathcal{V}_\infty$) quite different, but I will from time to time have occasion to refer to that work.[1]

Management of non-orthogonal bases: general principles. Let $\{|a_i)\}$ be linearly independent elements that span the real inner product space $\mathcal{V}_n$. In the absence of an orthogonality assumption, we write

$$(a_i|a_j) = g_{ij} \tag{1}$$

where $\|g_{ij}\|$ is real, symmetric and (by linear independence) non-singular. The

[1] "Reciprocal systems of non-orthogonal quantum states," (June, 1998).

generic element $|x)\in\mathcal{V}_n$ can be developed

$$|x) = |a_k)\,x^k \tag{2}$$

which gives

$$(a_j|x) = g_{jk}x^k$$

Writing $\|g_{ij}\|^{-1} = \|g^{ij}\|$ we have

$$g^{ij}(a_j|x) = g^{ij}g_{jk}x^k = \delta^i{}_k x^k = x^i \tag{3}$$

giving

$$|x) = |a_i)g^{ij}(a_j|x) \quad : \quad \text{all } |x)$$

from which we conclude that

$$|a_i)g^{ij}(a_j| = \mathbb{I} \tag{4}$$

Introduce now into $\mathcal{V}_n$ a second non-orthogonal "dual" basis with elements

$$|A^j) = |a_i)g^{ij} \quad \text{equivalently} \quad (A^i| = g^{ij}(a_j| \tag{5}$$

which supply this alternative constructiuon of the unit matrix

$$|a_i)(A^i| = \mathbb{I} \tag{6}$$

We have

$$(A^i|a_j) = g^{ik}(a_k|a_j) = g^{ik}g_{kj} = \delta^i{}_j \tag{7.1}$$

which is to say:

$$|A^i) \perp \text{all } |a_j) \quad : \quad j \neq i \tag{7.2}$$

It is on these grounds that the non-orthogonal bases $\{|a_i)\}$ and $\{|A^j)\}$ are said to be "biorthogonal" (or "reciprocal").[2]

Look now to the matrices

$$\mathbb{P}_i = |a_i)(A^i| \quad : \quad \text{no summation on } i \tag{8}$$

where the index placement on $\mathbb{P}_i$ is merely conventional (intended to convey no transformation-theoretic meaning). Those are seen to be orthogonal projection matrices

$$\mathbb{P}_i\mathbb{P}_j = |a_i)(A^i|a_j)(A^j| = |a_i)\delta^i{}_j(A^j| = \begin{cases} \mathbb{P}_i & : \quad i = j \\ \mathbb{O} & : \quad i \neq j \end{cases} \tag{9}$$

and were already seen at (6) to be complete: $\sum_i \mathbb{P}_i = \mathbb{I}$. That they project

[2] When $\{|a_i)\}$ is in fact orthonormal ($g^{ij} = \delta^{ij}$) the distinction between $\{|a_i)\}$ and $\{|A^i)\}$ evaporates.

onto 1-spaces (rays in $\mathcal{V}_n$) can be established as follows: let $\{|e_k)\}$ refer to any orthonormal basis in $\mathcal{V}_n$. Then

$$\begin{aligned}\mathrm{tr}\,\mathbb{P}_i = \textstyle\sum_k (e_k|\mathbb{P}_i|e_k) &= \textstyle\sum_k (e_k|a_i)(A^i|e_k)\\ &= \textstyle\sum_k (A^i|e_k)(e_k|a_i)\\ &= (A^i|a_i)\\ &= 1\end{aligned}\tag{10}$$

Specifically,

$$\left.\begin{aligned}\text{right action:}\quad \mathbb{P}_i|x) &= |a_i)\,x^i\\ \text{left action:}\quad (x|\mathbb{P}_i &= x_i\,(A^i|\end{aligned}\right\}\quad:\quad \text{no summation on } i$$

the point here being that for non-symmetric matrices we must distinguish between their right action and their left action (*i.e.*, between the action of the matrix and its transpose).

Given an arbitrary square matrix $\mathbb{M}$, we have

$$\begin{aligned}\mathbb{M} &= \mathbb{I}\,\mathbb{M}\,\mathbb{I}\\ &= \textstyle\sum_{ij} \mathbb{P}_i\,\mathbb{M}\,\mathbb{P}_j\\ &= \textstyle\sum_{ij} |a_i)(A^i|\mathbb{M}|a_j)(A^j|\\ &= \textstyle\sum_{ij} m^i{}_j |a_i)(A^j| \quad \text{where} \quad m^i{}_j = (A^i|\mathbb{M}|a_j)\end{aligned}\tag{11}$$

Here $\mathbb{M}$ is displayed as a weighted linear combination

$$\mathbb{M} = \textstyle\sum_{ij} m^i{}_j\, \mathbb{F}_i{}^j\tag{12}$$

of the n^2-member population of matrices

$$\mathbb{F}_i{}^j = |a_i)(A^j|\tag{13}$$

More particularly, $\|m^i{}_j\|$ provides the matrix representation of $\mathbb{M}$ with respect to the non-orthogonal $\{|a_i)\}$-basis; it permits $|x) \to |\tilde{x}) = \mathbb{M}|x)$ to be represented

$$x^i \to \tilde{x}^i = m^i{}_j x^j\tag{14}$$

From

$$\begin{aligned}\mathrm{tr}\left[\mathbb{F}_i{}^j\,\mathbb{F}_p{}^q\right] &= \textstyle\sum_k (e_k|a_i)(A^j|a_p)(A^q|e_k)\\ &= \textstyle\sum_k (A^q|e_k)(e_k|a_i)(A^j|a_p)\\ &= \delta_i{}^q (A^j|a_p)\\ &= \delta_i{}^q \delta^j{}_p\\ &= \begin{cases} 1 & : \quad q = i \text{ and } p = j\\ 0 & : \quad \text{otherwise}\end{cases}\end{aligned}$$

we see that the $\mathbb{F}$-matrices are tracewise orthogonal in the following sense:

$$\mathrm{tr}\big[\mathbb{F}_i{}^j\,\bar{\mathbb{F}}^q{}_p\big] = \begin{cases} 1 & : \quad \{i,j\} = \{q,p\} \\ 0 & : \quad \text{otherwise} \end{cases} \tag{15}$$

where

$$\bar{\mathbb{F}}^q{}_p = \mathbb{F}_p{}^q \tag{16}$$

It follows that if

$$\mathbb{M} = \textstyle\sum_{ij} m^i{}_j\,\mathbb{F}_i{}^j$$

then

$$m^i{}_j = \mathrm{tr}\big[\mathbb{M}\bar{\mathbb{F}}^i{}_j\big]$$

whence the "Fourier identity"[3]

$$\mathbb{M} = \textstyle\sum_{ij}\mathrm{tr}\big[\mathbb{M}\bar{\mathbb{F}}^i{}_j\big]\mathbb{F}_i{}^j \tag{17}$$

From

$$\mathbb{F}_i{}^p\,\mathbb{F}_q{}^j = \delta^p{}_q\,\mathbb{F}_i{}^j \tag{18}$$

we see that the set $\{F_i{}^j\}$ is multiplicatively closed, that $\mathbb{F}_i{}^p \perp \mathbb{F}_q{}^j$ unless $p = q$. The "diagonal members" $\mathbb{F}_i{}^i$ of $\{F_i{}^j\}$ are precisely the projective $\mathbb{P}$-matrices considered previously:

$$\mathbb{P}_i = \mathbb{F}_i{}^i \tag{19}$$

From (16) we have $\bar{\mathbb{P}}_i = \mathbb{P}_i$, while from (15) follows the tracewise orthogonality of those matrices:

$$\mathrm{tr}\big[\mathbb{P}_i\mathbb{P}_j\big] = \delta_{ij} \tag{20}$$

Generalized spectral decomposition. Assume—simply to keep simple things simple—that the eigenvalues of our otherwise arbitrary real square matrix $\mathbb{M}$ are real and distinct:[4]

$$\mathbb{M}|m_i) = \lambda_i|m_i) \tag{21}$$

Proceed as before to construct the elements

$$|M^j) = |m_i)g^{ij} \quad \text{equivalently} \quad (M^i| = g^{ij}(m_j|$$

of the dual $\{|M^i)\}$ of the eigenbasis $\{|m_i)\}$. The argument that led to (11) now supplies

$$\begin{aligned} \mathbb{M} &= \textstyle\sum_{ij}|m_i)(M^i|\mathbb{M}|m_j)(M^j| \\ &= \textstyle\sum_{ij}|m_i)\,\lambda_i\,\delta^i{}_j(M^j| \\ &= \textstyle\sum_i \lambda_i|m_i)(M^i| && (22.1) \\ &= \textstyle\sum_i \lambda_i\,\mathbb{P}_i && (22.2) \end{aligned}$$

[3] In all occurances of analogs of Fourier's identity one of the basis elements enters either conjugated or transposed or adjointed or otherwise goofy.

[4] Both assumptions are easily relaxed.

At (17) $\mathbb{M}$ was presented as a weighted sum of the n^2 elements of $\{\mathbb{F}_{ij}\}$, all but the "diagonal elements" of which[5] are non-projective. At (22.2) it is, on the other hand, presented as a weighted sum of the n elements of $\{\mathbb{P}_i\}$, *all* of which are projective.

Look to the special cases in which $\mathbb{M}$ is symmetric: $\mathbb{M}^{\mathsf{T}} = \mathbb{M}$. The reality of the eigenvalues is then automatic/assured, and so (if we assume spectral non-degeneracy) is the orthogonality of the eigenvectors. Assume without loss of generality that the eigenvectors have been normalized:

$$(m_i|m_j) = \delta_{ij}$$

Then $(M^i| = (m_i|$ and (22.1) becomes

$$\mathbb{M} = \textstyle\sum_i \lambda_i |m_i)(m_i|$$

which is familiar as the "spectral decomposition" of the symmetric matrix $\mathbb{M}$. Equations (22) demonstrate how the essential features/advantages of spectral decompositions can be preserved even when the familiar assumptions (symmetry/hermiticity/self-adjointness) have been abandoned.

Alternative constructions of the dual basis. In order once again to "keep simple things simple," to avoid notational clutter that would obscure the simple essence of the ideas at issue, we work in $\mathcal{V}_3$; how those ideas are to be realized in $\mathcal{V}_n$ will be obvious. Let the elements of an arbitrary unnormalized non-orthogonal basis in $\mathcal{V}_3$—denoted

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

be displayed as columns of a square matrix

$$\mathbb{B} = \|\,\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}\,\| = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

and let $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ refer to the rows of $\mathbb{B}^{-1}$:

$$\mathbb{B}^{-1} = \left\|\begin{matrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \end{matrix}\right\| = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

From $\mathbb{B}^{-1}\mathbb{B} = \mathbb{I}$ we see that the sets $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ and $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ reproduce the defining property $(A^i|a_j) = \delta^i{}_j$ of $\{|a_i)\}$ and $\{(A^i|\}$, which is to say: they are biorthogonal. Specifically (by *Mathematica*-assisted calculation)

---

[5] Those might be called the "self-conjugate" elements, since $\bar{\mathbb{F}}_i{}^j = \mathbb{F}_i{}^j$ if and only if $i = j$.

$$\begin{aligned}\boldsymbol{A} &= \frac{\boldsymbol{b}\times\boldsymbol{c}}{(\boldsymbol{abc})}\\ \boldsymbol{B} &= \frac{\boldsymbol{c}\times\boldsymbol{a}}{(\boldsymbol{abc})}\\ \boldsymbol{C} &= \frac{\boldsymbol{a}\times\boldsymbol{b}}{(\boldsymbol{abc})}\end{aligned}$$

where $(\boldsymbol{abc})$ denotes the triple scalar product

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \det\mathbb{B}$$

These results—which are, of course, specific to $\mathcal{V}_3$—make biorthogonality $(\boldsymbol{A}\perp\boldsymbol{b},\boldsymbol{c}\,;\ \boldsymbol{B}\perp\boldsymbol{c},\boldsymbol{a}\,;\ \boldsymbol{C}\perp\boldsymbol{a},\boldsymbol{b})$ obvious, and are standard to the physical/mathematical theories of chrystallographic lattices.[6]

From $\mathbb{B}^{-1}\mathbb{B}=\mathbb{I}\Rightarrow\mathbb{B}^{\mathsf{T}}(\mathbb{B}^{-1})^{\mathsf{T}}=\mathbb{I}$ we see that the dual of the dual basis is the original (or "direct") basis.

Alternatively, observe that

$$\mathbb{B}^{\mathsf{T}}\mathbb{B} = \begin{pmatrix}\boldsymbol{a}\cdot\boldsymbol{a} & \boldsymbol{a}\cdot\boldsymbol{b} & \boldsymbol{a}\cdot\boldsymbol{c}\\ \boldsymbol{b}\cdot\boldsymbol{a} & \boldsymbol{b}\cdot\boldsymbol{b} & \boldsymbol{b}\cdot\boldsymbol{c}\\ \boldsymbol{c}\cdot\boldsymbol{a} & \boldsymbol{c}\cdot\boldsymbol{b} & \boldsymbol{c}\cdot\boldsymbol{c}\end{pmatrix} = \|g_{ij}\|$$

gives

$$(\mathbb{B}^{\mathsf{T}}\mathbb{B})^{-1} = \begin{pmatrix}g^{aa} & g^{ab} & g^{ac}\\ g^{ba} & g^{bb} & g^{bc}\\ g^{ca} & g^{cb} & g^{cc}\end{pmatrix}$$

whence[7]

$$\begin{aligned}\mathbb{B}(\mathbb{B}^{\mathsf{T}}\mathbb{B})^{-1} &= \|\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}\|\begin{pmatrix}g^{aa} & g^{ab} & g^{ac}\\ g^{ba} & g^{bb} & g^{bc}\\ g^{ca} & g^{cb} & g^{cc}\end{pmatrix}\\ &= \|(\boldsymbol{a}g^{aa}+\boldsymbol{b}g^{ba}+\boldsymbol{c}g^{ca})\quad(\boldsymbol{a}g^{ab}+\boldsymbol{b}g^{bb}+\boldsymbol{c}g^{cb})\quad(\boldsymbol{a}g^{ac}+\boldsymbol{b}g^{bc}+\boldsymbol{c}g^{cc})\|\\ &= \|\boldsymbol{A}\ \boldsymbol{B}\ \boldsymbol{C}\|\\ &= (\mathbb{B}^{-1})^{\mathsf{T}}\equiv\mathbb{A}\end{aligned}$$

We see that a matrix inversion—whether of $\|g_{ij}\|$, $\mathbb{B}$ or $\mathbb{B}^{\mathsf{T}}\mathbb{B}$—is required however one proceeds to obtain the elements of the dual basis.

Basis transformations. Let $\{|b_i)\}$ and $\{|B^i)\}$ refer to a second non-orthogonal basis and its dual. Writing $(b_i|b_j)=\bar{g}_{ij}$ we have as before $(B^i|B^j)=\bar{g}^{ij}$, $(b_i|B^j)=(B^j|b_i)=\delta^j{}_i$ and $\sum_i|b_i)(B^i|=\sum_i|B^i)(b_i|=\mathbb{I}$. We are placed in

[6] See §3, pages 5–10 in the essay previously mentioned,[1] and references cited there.

[7] See again (5).

position therefore to write

$$
\begin{aligned}
|x) = |a_i)x^i = |b_i)\tilde{x}^i \\
\tilde{x}^i &= (B^i|x) = (B^i|a_j)(A^j|x) \\
&= (B^i|a_j)x^j \\
&= S^i{}_j x^j \quad : \quad S^i{}_j \equiv (B^i|a_j) \\
\tilde{\boldsymbol{x}} &= \mathbb{S}\,\boldsymbol{x}
\end{aligned}
\tag{23.1}
$$

Similarly

$$
\begin{aligned}
\tilde{m}^i{}_j = (B^i|\mathbb{M}|b_j) &= (B^i|a_p)(A^p|\mathbb{M}|a_q)(A^q|b_j) \\
&= S^i{}_p m^p{}_q T^q{}_j \quad : \quad T^i{}_j \equiv (A^i|b_j) \\
\tilde{\mathbb{M}} &= \mathbb{S}\,\mathbb{M}\,\mathbb{T}
\end{aligned}
$$

From $\mathbb{S}\,\mathbb{T} = \|(B^i|a_k)(A^k|b_j)\| = \|(B^i|b_j)\| = \|\delta^i{}_j\| = \mathbb{I}$ we have $\mathbb{T} = \mathbb{S}^{-1}$, so $\mathbb{M} \to \tilde{\mathbb{M}}$ by similarity transformation

$$
\tilde{\mathbb{M}} = \mathbb{S}\,\mathbb{M}\,\mathbb{S}^{-1} \tag{23.2}
$$

from which it follows in particular that $\mathbb{M}$ and $\tilde{\mathbb{M}}$ have identical spectra.

The projection matrices $\mathbb{P}_i$ that appear in the spectral decomposition (22) of $\mathbb{M}$ were assembled from (designed to project onto) the eigenrays of $\mathbb{M}$. Looking to the arbitrary transform of $\mathbb{M}\,\boldsymbol{m}_i = \lambda_i\,\boldsymbol{m}_i$ we have

$$
\mathbb{S}\,\mathbb{M}\,\mathbb{S}^{-1}\cdot\mathbb{S}\,\boldsymbol{m}_i = \lambda_i \mathbb{S}\,\boldsymbol{m}_i \quad : \quad \tilde{\mathbb{M}}\,\tilde{\boldsymbol{m}}_i = \lambda_i\,\tilde{\boldsymbol{m}}_i \quad \text{with} \quad \tilde{\boldsymbol{m}}_i = \mathbb{S}\,\boldsymbol{m}_i
$$

which states simply that "being an eigenvector of" refers to a representation-independent state of affairs. So also does the spectral decomposition:

$$
\tilde{\mathbb{M}} = \textstyle\sum_i \lambda_i\,\tilde{\mathbb{P}}_i \quad \text{with} \quad \tilde{\mathbb{P}}_i = \mathbb{S}\,\mathbb{P}_i\,\mathbb{S}^{-1}
$$

We observe finally that

$$
\bar{g}_{ij} \equiv (b_i|b_j) = (b_i|A^p)(a_p|a_q)(A^q|b_j) = (b_i|A^p)g_{pq}(A^q|b_j)
$$

gives

$$
\|\bar{g}_{ij}\| = \mathbb{T}^{\mathsf{T}}\|g_{pq}\|\,\mathbb{T} \tag{23.3}
$$

2-dimensional generalities. As was remarked at the outset, it is as it relates to $\mathcal{V}_2$ that the material discussed above acquires direct relevance to the differential geometry of surfaces. We look now, in a general way, to the specifics of the 2-dimensional case, taking full advantage of the simplifications that become available in that case; those have to do with the circumstance that all general results can be made notationally explicit, and with the ease with which square matrices can in two dimensions be inverted.

Let

$$|a_1) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad |a_2) = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \tag{24.1}$$

refer to an arbitrary unnormalized non-orthogonal basis in $\mathcal{V}_2$. Then

$$\left.\begin{aligned} g_{11} &= \alpha_1^2 + \alpha_2^2 \\ g_{12} = g_{21} &= \alpha_1\beta_1 + \alpha_2\beta_2 \\ g_{22} &= \beta_1^2 + \beta_2^2 \end{aligned}\right\} \tag{24.2}$$

$$\mathbb{G} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{24.3}$$

$$g \equiv \det \mathbb{G} = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 \tag{24.4}$$

$$\mathbb{G}^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = g^{-1} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} \tag{24.5}$$

$$\left.\begin{aligned} |A^1) &= |a_1)g^{11} + |a_2)g^{21} = \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\begin{pmatrix} \beta_2 \\ -\beta_1 \end{pmatrix} \\ |A^2) &= |a_1)g^{12} + |a_2)g^{22} = \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\begin{pmatrix} -\alpha_2 \\ \alpha_1 \end{pmatrix} \end{aligned}\right\} \tag{24.6}$$

The construction (24.6) of the basis $\{|A^i)\}$ dual to $\{|a_i)\}$—obtained here by specialization of (5)—is in fact an immediately forced direct consequence of the biorthogonality conditions $|A^1) \perp |a_2), |A^2) \perp |a_1), (A^1|a_1) = (A^2|a_2) = 1$. The labor that went into the construction of (24.6) was therefore hardly necessary, but yielded a result that (as we verify) does conform to

$$(A^i|a_j) = \delta^i{}_j \tag{24.7}$$

With *Mathematica*'s assistance we verify that the matrices

$$\left.\begin{aligned} \mathbb{P}_1 &= \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\begin{pmatrix} \alpha_1\beta_2 & -\alpha_1\beta_1 \\ \alpha_2\beta_2 & -\alpha_2\beta_1 \end{pmatrix} \\ \mathbb{P}_2 &= \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\begin{pmatrix} -\alpha_2\beta_1 & \alpha_1\beta_1 \\ -\alpha_2\beta_2 & \alpha_1\beta_2 \end{pmatrix} \end{aligned}\right\} \tag{25.1}$$

comprise a complete

$$\mathbb{P}_1 + \mathbb{P}_2 = \mathbb{I} \tag{25.21}$$

set of orthogonal

$$\mathbb{P}_1\mathbb{P}_2 = \mathbb{P}_2\mathbb{P}_1 = \mathbb{O} \tag{25.22}$$

projection matrices

$$\mathbb{P}_1\mathbb{P}_1 = \mathbb{P}_1, \quad \mathbb{P}_2\mathbb{P}_2 = \mathbb{P}_2 \tag{25.23}$$

and that they are trace-wise orthogonal:

$$\mathrm{tr}\big[\mathbb{P}_i\mathbb{P}_j\big] = \delta_{ij} \tag{25.3}$$

Given an arbitrary element $|x)\in\mathcal{V}_2$

$$|x) = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{26.1}$$

we have

$$|x) = |a_1)x^1 + |a_2)x^2 \quad \text{with} \quad \begin{cases} x^1 = (A^1|x) = -\dfrac{\beta_1\xi_2 - \beta_2\xi_1}{\alpha_1\beta_2 - \alpha_2\beta_1} \\[2ex] x^2 = (A^2|x) = \dfrac{\alpha_1\xi_2 - \alpha_2\xi_1}{\alpha_1\beta_2 - \alpha_2\beta_1} \end{cases} \tag{26.2}$$

Given an arbitrary real square matrix

$$\mathbb{M} = \begin{pmatrix} a & b \\ d & c \end{pmatrix} \qquad (27.1$$

we by (11) have

$$\mathbb{M} = m^1{}_1\mathbb{F}_1{}^1 + m^1{}_2\mathbb{F}_1{}^2 + m^2{}_1\mathbb{F}_2{}^1 + m^2{}_2\mathbb{F}_2{}^2 \quad : \quad \mathbb{F}_i{}^j \equiv |a_i)(A^j| \tag{27.2}$$

where

$$\left.\begin{aligned} \mathbb{F}_1{}^1 &= \mathbb{P}_1 \\ \mathbb{F}_1{}^2 &= \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\begin{pmatrix} -\alpha_1\alpha_2 & \alpha_1\alpha_1 \\ -\alpha_2\alpha_2 & \alpha_2\alpha_1 \end{pmatrix} \\ \mathbb{F}_2{}^1 &= \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\begin{pmatrix} \beta_1\beta_2 & -\beta_1\beta_1 \\ \beta_2\beta_2 & -\beta_2\beta_1 \end{pmatrix} \\ \mathbb{F}_2{}^2 &= \mathbb{P}_2 \end{aligned}\right\} \tag{27.3}$$

are—as *Mathematica* confirms—tracewise orthogonal in the sense that

$$\mathrm{tr}\big[\mathbb{F}_i{}^j\mathbb{F}_p{}^q\big] = \delta_i{}^q\delta^j{}_p \tag{27.4}$$

By computation

$$\begin{aligned} m^1{}_1 &= (A^1|\mathbb{M}|a_1) = \mathrm{tr}\big[\mathbb{M}\mathbb{F}_1{}^1\big] = g^{-\frac{1}{2}}\big[\quad \alpha_1(a\beta_2 - d\beta_1) - \alpha_2(c\beta_1 - b\beta_2)\big] \\ m^1{}_2 &= (A^1|\mathbb{M}|a_2) = \mathrm{tr}\big[\mathbb{M}\mathbb{F}_2{}^1\big] = g^{-\frac{1}{2}}\big[(a-c)\,\beta_1\beta_2 + b\,\beta_2\beta_2 - d\,\beta_1\beta_1\big] \\ m^2{}_1 &= (A^2|\mathbb{M}|a_1) = \mathrm{tr}\big[\mathbb{M}\mathbb{F}_1{}^2\big] = g^{-\frac{1}{2}}\big[(c-a)\alpha_1\alpha_2 - b\alpha_2\alpha_2 + d\alpha_1\alpha_1\big] \\ m^2{}_2 &= (A^2|\mathbb{M}|a_2) = \mathrm{tr}\big[\mathbb{M}\mathbb{F}_2{}^2\big] = g^{-\frac{1}{2}}\big[-\beta_1(a\alpha_2 - d\alpha_1) + \beta_2(c\alpha_1 - b\alpha_2)\big] \end{aligned}$$

which check out when introduced into (27.2), so we have the Fourier expansion

$$\mathbb{M} = \mathrm{tr}\big[\mathbb{M}\mathbb{F}_1{}^1\big]\mathbb{F}_1{}^1 + \mathrm{tr}\big[\mathbb{M}\mathbb{F}_2{}^1\big]\mathbb{F}_1{}^2 + \mathrm{tr}\big[\mathbb{M}\mathbb{F}_1{}^2\big]\mathbb{F}_2{}^1 + \mathrm{tr}\big[\mathbb{M}\mathbb{F}_2{}^2\big]\mathbb{F}_2{}^2$$

We verify also that

$$\det\begin{pmatrix} m^1{}_1 & m^1{}_2 \\ m^2{}_1 & m^2{}_2 \end{pmatrix} = ac - bd = \det\mathbb{M}$$

It is interesting in this light to notice that setting $d = b$ symmetrizes $\mathbb{M}$ but does not symmetrize $\|m^i{}_j\|$.

Reformulations for geometric applications. We take now into account the fact that the matrices $\mathbb{G}$ and $\mathbb{H}$ of differential geometric interest are *symmetric*, and (to minimize the number of free parameters) exercise our option to work only witn *normalized* vectors.

Let

$$|a_1) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \quad |a_2) = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} \tag{28.1}$$

refer to an arbitrary normalized basis in $\mathcal{V}_2$. Immediately

$$\|g_{ij}\| = \begin{pmatrix} 1 & \cos(\alpha-\beta) \\ \cos(\alpha-\beta) & 1 \end{pmatrix} \quad \text{and} \quad \det\|g_{ij}\| = \sin^2(\alpha-\beta)$$

The elements of the dual basis could be obtained from (5) or by either of the (equivalent) methods described on pages 5–6, but it is easiest to proceed from the requirements that $|A^1) \perp |a_2)$, $(A^1|a_1) = 1$ and $|A^2) \perp |a_1)$, $(A^2|a_2) = 1$, which give

$$\begin{aligned} |A^1) &= k\begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix}, \quad k = +\csc(\alpha-\beta) \\ |A^2) &= k\begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}, \quad k = -\csc(\alpha-\beta) \end{aligned}$$

whence

$$|A^1) = \csc(\alpha-\beta)\begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix}, \quad |A^2) = \csc(\alpha-\beta)\begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix} \tag{28.2}$$

By *Mathematica*-assisted computation we find from $\mathbb{F}_i{}^j = |a_i)(A^j|$ that

$$\left.\begin{aligned} \mathbb{P}_1 = \mathbb{F}_1{}^1 &= \csc(\alpha-\beta)\begin{pmatrix} -\cos\alpha\sin\beta & \cos\alpha\cos\beta \\ -\sin\alpha\sin\beta & \sin\alpha\cos\beta \end{pmatrix} \\ \mathbb{F}_1{}^2 &= \csc(\alpha-\beta)\begin{pmatrix} \cos\alpha\sin\alpha & -\cos^2\alpha \\ \sin^2\alpha & -\cos\alpha\sin\alpha \end{pmatrix} \\ \mathbb{F}_2{}^1 &= \csc(\alpha-\beta)\begin{pmatrix} -\cos\beta\sin\beta & \cos^2\beta \\ -\sin^2\beta & \cos\beta\sin\beta \end{pmatrix} \\ \mathbb{P}_2 = \mathbb{F}_2{}^2 &= \csc(\alpha-\beta)\begin{pmatrix} \cos\beta\sin\alpha & -\cos\beta\cos\alpha \\ \sin\beta\sin\alpha & -\sin\beta\cos\alpha \end{pmatrix} \end{aligned}\right\} \tag{28.3}$$

and verify that those matrices satisfy the trace relations (27.4), also that $\{\mathbb{P}_1, \mathbb{P}_2\}$ comprise a complete orthogonal set of tracewise-orthonormal projection matrices.

Look now to the symmetric matrix

$$\mathbb{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{29.1}$$

From the characteristic polynomial

$$\det(\mathbb{M} - \lambda\mathbb{I}) = \lambda^2 - \lambda\,\mathrm{tr}\,\mathbb{M} + \det\mathbb{M}$$

we obtain eigenvalues

$$\begin{aligned}\lambda_\pm &= \tfrac{1}{2}\Big[\mathrm{tr}\,\mathbb{M} \pm \sqrt{\mathrm{tr}^2\,\mathbb{M} - \det\mathbb{M}}\,\Big]\\ &= \tfrac{1}{2}\Big[(a+c) \pm \sqrt{(a-c)^2 + 4b^2}\,\Big]\end{aligned}\tag{29.2}$$

(henceforth denoted λ_1 and λ_2, respectively) that are manifestly real, and that are of the same or opposite signs according as $\det\mathbb{M} = \lambda_1\lambda_2 = ac - b^2$ is $\gtrless 0$. These results will come into play shortly.

The associated eigenvectors are well known to be (in the absence of spectral degeneracy) invariably orthogonal, and—since assumed to have been normalized—can be parameterized

$$|m_1) = \begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \quad |m_2) = \begin{pmatrix}-\sin\phi\\ \cos\phi\end{pmatrix} = |m_1)\Big|_{\phi\to\phi+\frac{1}{2}\pi}\tag{29.31}$$

Orthonormality means $g_{ij} = \delta_{ij}$, $g^{ij} = \delta^{ij}$, so the distinction between the basis and its dual evaporates[2]

$$|M^1) = |m_1), \quad |M^2) = |m_2)\tag{29.32}$$

and from $\mathbb{F}_i{}^j = |m_i)(M^j|$ we obtain

$$\left.\begin{aligned}\mathbb{P}_1 = \mathbb{F}_1{}^1 &= \begin{pmatrix}\cos^2\phi & \cos\phi\sin\phi\\ \cos\phi\sin\phi & \sin^2\phi\end{pmatrix}\\ \mathbb{F}_1{}^2 &= \begin{pmatrix}-\cos\phi\sin\phi & \cos^2\phi\\ -\sin^2\phi & \cos\phi\sin\phi\end{pmatrix}\\ \mathbb{F}_2{}^1 &= \begin{pmatrix}-\cos\phi\sin\phi & -\sin^2\phi\\ \cos^2\phi & \cos\phi\sin\phi\end{pmatrix}\\ \mathbb{P}_2 = \mathbb{F}_2{}^2 &= \begin{pmatrix}\sin^2\phi & -\cos\phi\sin\phi\\ -\cos\phi\sin\phi & \cos^2\phi\end{pmatrix}\end{aligned}\right\}\tag{29.4}$$

and verify that those matrices do possess all the anticipated properties. By computation

$$\left.\begin{aligned}m^1{}_1 &= (M^1|\mathbb{M}|m_1) = a\cos^2\phi + 2b\cos\phi\sin\phi + c\sin^2\phi\\ m^1{}_2 &= (M^1|\mathbb{M}|m_2) = b\cos2\phi - \tfrac{1}{2}(a-c)\sin2\phi\\ m^2{}_1 &= (M^2|\mathbb{M}|m_1) = b\cos2\phi - \tfrac{1}{2}(a-c)\sin2\phi\\ m^2{}_2 &= (M^2|\mathbb{M}|m_2) = a\sin^2\phi - 2b\cos\phi\sin\phi + c\cos^2\phi\end{aligned}\right\}\tag{29.5}$$

We observe that these results could have been obtained from

$$\begin{pmatrix}m^1{}_1 & m^1{}_2\\ m^2{}_1 & m^2{}_2\end{pmatrix} = \mathbb{R}\mathbb{M}\mathbb{R}^{-1} \text{ where } \mathbb{R} \text{ is the rotation matrix } \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}$$

The normalized eigenvectors of $\mathbb{M}$ emerge when ϕ is assigned a value that diagonalizes $\|m^i{}_j\|$. From

$$m^1{}_2 = m^2{}_1 = b\cos 2\phi - \tfrac{1}{2}(a-c)\sin 2\phi = 0$$

we obtain

$$\phi = \tfrac{1}{2}\arctan\Big[\frac{2b}{a-c}\Big] \tag{29.6}$$

which with *Mathematica*'s assistance is found to give[8]

$$\begin{aligned} m^1{}_1 &= \lambda_1 \\ m^2{}_2 &= \lambda_2 \end{aligned}$$

From (29.6) we obtain[8]

$$\left.\begin{aligned} \cos\phi &= \sqrt{\tfrac{1}{2}\Big(1+\frac{a-c}{\sqrt{D}}\Big)} \\ \sin\phi &= \sqrt{\tfrac{1}{2}\Big(1-\frac{a-c}{\sqrt{D}}\Big)} \end{aligned}\right\} \quad : \quad D = (a-c)^2 + 4b^2 \tag{29.7}$$

and by tedious calculation verify that

$$\begin{aligned} &\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \\ &\qquad = \begin{pmatrix} \frac{1}{2}\big[(a+c)+\sqrt{D}\,\big] & 0 \\ 0 & \frac{1}{2}\big[(a+c)-\sqrt{D}\,\big] \end{pmatrix} \end{aligned} \tag{29.8}$$

Bringing (29.7) to (29.4) we obtain

$$\left.\begin{aligned} \mathbb{P}_1 &= \begin{pmatrix} \frac{1}{2}\Big(1+\frac{a-c}{\sqrt{D}}\Big) & \frac{1}{2}\Big(1-\frac{(a-c)^2}{D}\Big) \\ \frac{1}{2}\Big(1-\frac{(a-c)^2}{D}\Big) & \frac{1}{2}\Big(1-\frac{a-c}{\sqrt{D}}\Big) \end{pmatrix} \\ \mathbb{P}_2 &= \begin{pmatrix} \frac{1}{2}\Big(1-\frac{a-c}{\sqrt{D}}\Big) & -\frac{1}{2}\Big(1-\frac{(a-c)^2}{D}\Big) \\ -\frac{1}{2}\Big(1-\frac{(a-c)^2}{D}\Big) & \frac{1}{2}\Big(1+\frac{a-c}{\sqrt{D}}\Big) \end{pmatrix} \end{aligned}\right\} \tag{29.9}$$

and verify that

$$\mathbb{M} = \tfrac{1}{2}\big[(a+c)+\sqrt{D}\,\big]\mathbb{P}_1 + \tfrac{1}{2}\big[(a+c)-\sqrt{D}\,\big]\mathbb{P}_2 \tag{29.10}$$

Which is far more than any sane person could possibly want to know about the spectral theory of real 2×2 symmetric matrices, and is of conceivable interest only as it relates comparatively to the topic to which I now turn.

---

[8] Use `TrigToExp` and `Simplify`.

Look to the conditions under which the normalized vector

$$|a) = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$$

satisfies

$$(a|\mathbb{M}|a) = 0 \tag{30.1}$$

Immediately

$$\tan\psi = \frac{-b \pm \sqrt{b^2 - ac}}{c} \tag{30.2}$$

which is real if and only if $\det\mathbb{M} = ac - b^2 < 0$.[9] Such "asymptotic" vectors occur, therefore, in conjugate pairs

$$|a_1) = \begin{pmatrix} \cos\psi_1 \\ \sin\psi_1 \end{pmatrix}, \quad |a_2) = \begin{pmatrix} \cos\psi_2 \\ \sin\psi_2 \end{pmatrix} \tag{30.3}$$

where ψ_1 and ψ_2 are given by

$$\psi_\pm = \arctan\Big[\frac{-b \pm \sqrt{b^2 - ac}}{c}\Big] \tag{30.4}$$

respectively. From

$$\|g_{ij}\| = \|(a_i|a_j)\| = \begin{pmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{pmatrix} \tag{30.5}$$

$$\cos\omega = (a_1|a_2) = \cos(\psi_1 - \psi_2)$$
$$\omega = \text{subtended angle}$$

we see that $\{|a_1), |a_2)\}$ comprises a typically[10] non-orthogonal basis in $\mathcal{V}_2$. We look therefore to the construction of the dual basis $\{|A^1), |A^2)\}$. From (30.5) we have

$$\|g^{ij}\| = \csc^2\omega \begin{pmatrix} 1 & -\cos\omega \\ -\cos\omega & 1 \end{pmatrix}$$

so (5)—after simplifications—gives

$$|A^1) = \csc\omega \begin{pmatrix} -\sin\psi_2 \\ \cos\psi_2 \end{pmatrix}, \quad |A^2) = \csc\omega \begin{pmatrix} \sin\psi_1 \\ -\cos\psi_1 \end{pmatrix} \tag{30.6}$$

---

[9] We exclude the uninteresting case $ac - b^2 = 0$.

[10] From arctan y = arctan x $\pm \frac{1}{2}\pi$ we obtain $y = -1/x$, so $\psi_2 = \psi_1 \pm \frac{1}{2}\pi$ reads

$$\frac{-b - \sqrt{b^2 - ac}}{c} = -\frac{c}{-b + \sqrt{b^2 - ac}}$$

which entails $(a + c)c = 0$. We verify that the eigenvectors of $\mathbb{M}$ are in fact orthogonal if either $a = -c$ or $c = 0$. In the former case the eigenvalues are $\lambda_\pm = \pm\sqrt{b^2 + c^2}$, in the latter case $\lambda_\pm = \pm b$; in both cases they differ only by sign.

Working from (8), we find

$$\left.\begin{aligned}\mathbb{P}_1 = |a_1)(A^1| &= \csc\omega\begin{pmatrix}-\cos\psi_1\sin\psi_2 & \cos\psi_1\cos\psi_2\\ -\sin\psi_1\sin\psi_2 & \sin\psi_1\cos\psi_2\end{pmatrix}\\ \mathbb{P}_2 = |a_2)(A^2| &= \csc\omega\begin{pmatrix}\cos\psi_2\sin\psi_1 & -\cos\psi_2\cos\psi_1\\ \sin\psi_2\sin\psi_1 & -\sin\psi_2\cos\psi_1\end{pmatrix}\end{aligned}\right\}\tag{30.7}$$

and verify that these do in fact comprise a complete set of orthogonal projection matrices, and moreover that they possess the anticipated trace relations; *i.e.*, that they are trace-wise orthonormal:

$$\mathrm{tr}\big[\mathbb{P}_i\mathbb{P}_j\big] = \delta_{ij}\tag{30.8}$$

With *Mathematica*'s aid we find that the elements $m^i{}_j = (A^i|\mathbb{M}|a_j)$ of the asymptotic representation of $\mathbb{M}$ are given by

$$\begin{aligned}\sin\omega\cdot m^1{}_1 &= -a\,\cos\psi_1\sin\psi_2 + b\,\cos(\psi_1+\psi_2) + c\,\cos\psi_2\sin\psi_1\\ \sin\omega\cdot m^2{}_2 &= a\,\cos\psi_2\sin\psi_1 - b\,\cos(\psi_1+\psi_2) - c\,\cos\psi_1\sin\psi_2\\ \sin\omega\cdot m^1{}_2 &= -\tfrac{1}{2}(a-c)\,\sin 2\psi_2 + b\,\cos 2\psi_2\\ \sin\omega\cdot m^2{}_1 &= \tfrac{1}{2}(a-c)\,\sin 2\psi_1 - b\,\cos 2\psi_1\end{aligned}\tag{30.9}$$

To express these results in terms of $\{a,b,c\}$ we bring into play the identities

$$\cos[\arctan x] = \frac{1}{\sqrt{1+x^2}}\,,\quad \sin[\arctan x] = \frac{x}{\sqrt{1+x^2}}$$

of which

$$\begin{aligned}\cos[2\arctan x] &= \frac{1-x^2}{1+x^2}\\ \sin[2\arctan x] &= \frac{2x}{1+x^2}\\ \cos[\arctan x + \arctan y] &= \frac{1-xy}{\sqrt{1+x^2}\sqrt{1+y^2}}\\ \sin[\arctan x - \arctan y] &= \frac{x-y}{\sqrt{1+x^2}\sqrt{1+y^2}}\end{aligned}$$

are corollaries. Let (30.4) be abbreviated $\psi_1 = \tan^{-1}x$, $\psi_2 = \tan^{-1}y$. We then have

$$\begin{aligned}m^1{}_1 &= \frac{-ay + b(1-xy) + cx}{x-y}\\ m^2{}_2 &= \frac{ax - b(1-xy) - cy}{x-y}\\ m^1{}_2 &= \frac{-(a-c)y + b(1-y^2)}{x-y}\sqrt{\frac{1+x^2}{1+y^2}}\\ m^2{}_1 &= \frac{(a-c)x - b(1-x^2)}{x-y}\sqrt{\frac{1+y^2}{1+x^2}}\end{aligned}$$

Assigning to $\{x, y\}$ the values

$$x = \frac{-b + \sqrt{b^2 - ac}}{c}, \quad y = \frac{-b - \sqrt{b^2 - ac}}{c}$$

taken from (30.2), we obtain (after some fairly heroic simplification)

$$\left.\begin{aligned} m^1{}_1 &= \tfrac{1}{2}(a+c) \\ m^2{}_2 &= \tfrac{1}{2}(a+c) \\ m^1{}_2 &= -\tfrac{1}{2}\sqrt{(a-c)^2 + 4b^2} \\ m^2{}_1 &= -\tfrac{1}{2}\sqrt{(a-c)^2 + 4b^2} \end{aligned}\right\} \tag{30.10}$$

From these results we have

$$\begin{aligned} \mathrm{tr}\,\|m^i{}_j\| &= a + c \quad\;\; = \mathrm{tr}\,\mathbb{M} \\ \det\|m^i{}_j\| &= ac - b^2 = \det\mathbb{M} \end{aligned} \tag{30.11}$$

which conform to the observation[11] that basis transformations—in this instance

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \longrightarrow \left\{ |a_1) = \begin{pmatrix} \cos\psi_1 \\ \sin\psi_1 \end{pmatrix}, |a_2) = \begin{pmatrix} \cos\psi_2 \\ \sin\psi_2 \end{pmatrix} \right\}$$

—are accomplished invariably by similarity transformations:

$$\|m^i{}_j\| = \mathbb{S}\mathbb{M}\mathbb{S}^{-1}$$

From (23.1) and (30.6) we obtain

$$\mathbb{S} = \csc\omega \begin{pmatrix} -\sin\psi_2 & \cos\psi_2 \\ \sin\psi_1 & -\cos\psi_1 \end{pmatrix}, \quad \mathbb{S}^{-1} = \begin{pmatrix} \cos\psi_1 & \cos\psi_2 \\ \sin\psi_1 & \sin\psi_2 \end{pmatrix}$$

and from $\mathbb{S}\mathbb{M}\mathbb{S}^{-1}$ are gratified to recover precisely (30.9). The result just established—which can be summarized

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \xrightarrow[\text{asymptotic representation}]{\hspace{8em}} \begin{pmatrix} p & q \\ q & p \end{pmatrix} \tag{31}$$

with $p = \frac{1}{2}(a+c)$, $q = -\frac{1}{2}\sqrt{(a-c)^2 + 4b^2}$ —possesses a certain elegant charm, since $\{p, q\}$ were encountered already at (29.2) as fragments of the (shared) eigenvalues

$$\lambda_\pm = p \mp q$$

[11] See again (23.2) on page 7.

The following special cases are of some interest:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \xrightarrow[b=0]{\hspace{4em}} \begin{pmatrix} \frac{1}{2}(a+c) & -\frac{1}{2}(a-c) \\ -\frac{1}{2}(a-c) & \frac{1}{2}(a+c) \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \xrightarrow[c=0]{\hspace{4em}} \begin{pmatrix} \frac{1}{2}a & -\frac{1}{2}\sqrt{a^2+4b^2} \\ -\frac{1}{2}\sqrt{a^2+4b^2} & \frac{1}{2}a \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \xrightarrow[c=a]{\hspace{4em}} \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \xrightarrow[c=-a]{\hspace{4em}} \begin{pmatrix} 0 & -\sqrt{a^2+b^2} \\ -\sqrt{a^2+b^2} & 0 \end{pmatrix} \tag{$\star$}$$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \xrightarrow[b^2=ac]{\hspace{4em}} \begin{pmatrix} \frac{1}{2}(a+c) & -\frac{1}{2}(a+c) \\ -\frac{1}{2}(a+c) & \frac{1}{2}(a+c) \end{pmatrix}$$

All of those asymptotic matrices exhibit the distinctive symmetry structure of (31). As they must, but which is nevertheless a bit surprising. For we cannot generally expect similarity transformations (recall (23.2)) to preserve symmetry.[12] The symmetry of the asymptotic representation $\|m^i{}_j\|$ of $\mathbb{M}$, which emerged at (30.10), derives from the circumstance that the structure of $\mathbb{S}$ is "tuned" to conform to that of $\mathbb{M}$, in the sense that its elements are assembled in a very paticular way from those of $\mathbb{M}$.

Concluding remarks. Early on in their discussion of Tzitzeica surfaces,[13] Rogers & Schief remark that "hyperbolic surfaces Σ can always be parameterized in terms of real asymptotic coordinates. In this case [meaning unspecified], the Gauss equations take the form. . . " whereupon they write a version of the Gauss equations in which the parameters e and g (diagonal elements of $\mathbb{H}$) have been set equal to zero. I undertook this little project to discover what "this case" might mean, my initial hunch being that in asymptotic coordinates the diagonal elements *invariably* vanish. That conjecture is contradicted (except in the case marked $\star$) by the preceding examples, and was anyway shown to be untenable when at (23.2) it was established that the trace is invariant under *all* basis transformations. For certainly it is not the case that $\mathrm{tr}\,\mathbb{H} = 0$ holds universally.

The only Tzitzeica surface presently known to me is H. Jonas' "hexenhut,"

$$z(x^2+y^2) = \alpha^2 \quad : \quad \alpha^2 = \tfrac{2}{3\sqrt{3}}$$

[12] Symmetry of $\mathbb{M} \Longrightarrow$ symmetry of $\mathbb{S}\mathbb{M}\mathbb{S}^{-1}$ if and only if $\mathbb{S}^{-1} = k\mathbb{S}^{\mathsf{T}}$.

[13] *Bäcklund and Darboux Transformations: Geometry & Modern Applications in Soliton Theory* (2002), Chapter 3, page 89.

which is a surface of revolution with everywhere negative curvature (see Rogers & Schief, pages 105–106). In "Geodesics on the Pseudosphere & Hexenhut" (January 2016) I looked to the asymptotic parameterization of the hexenhut, and found

$$\left.\begin{aligned} h_{11} = h_{22} &= \alpha^2\frac{4\beta^2-3}{32\beta^3}\cdot\frac{1}{\sqrt{\boldsymbol{n\cdot n}}} \\ h_{12} = h_{21} &= \alpha^2\frac{4\beta^2+3}{32\beta^3}\cdot\frac{1}{\sqrt{\boldsymbol{n\cdot n}}} \end{aligned}\right\} \quad : \quad \beta = \tfrac{\sqrt{3}}{2}$$

but neglected to notice that the stated value of β implies $h_{11} = h_{22} = 0$; *i.e.*,

$$\mathbb{H}_{\text{asymptotic}} = \begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix} \quad \text{with} \quad f = \tfrac{2}{9}\frac{1}{\sqrt{\boldsymbol{n\cdot n}}}$$

But that attractive result raises a fresh question. For when I worked from the "natural" parameterization

$$\boldsymbol{r} = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ u \end{pmatrix} \quad \text{with} \quad f(u) = \alpha/\sqrt{u}$$

I found

$$\mathbb{H}_{\text{natural}} = \begin{pmatrix} e & 0 \\ 0 & g \end{pmatrix} \quad \text{with} \quad \begin{cases} e = -\dfrac{3\alpha}{2u\sqrt{4u^3+\alpha^2}} \\ g = \dfrac{2u\alpha}{\sqrt{4u^3+\alpha^2}} \end{cases}$$

which, though symmetric, is *not* traceless. I presently understand this seeming contradiction as evidence that it is mistake—at least so far as concerns $\mathbb{H}$, into which *second* derivatives enter in an essential way—to conflate parameter transformations and basis transformations. I will take up this issue in a companion essay.